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RESEARCH ARTICLE

The magnetisation of two ferromagnetic spheres placed randomly in a magnetic field

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Abstract: The main objective of this research is to study the uniform magnetic field both inside and outside of two randomly placed ideal ferromagnetic spheres. The expressions are given for the scalar magnetic potentials both inside and outside the spheres. In addition, the expressions for the magnetic field intensity outside and the magnetic flux density inside are also derived. The magnetic potential outside the spheres is obtained by the superposition of the potentials due to the two spheres and the potential corresponding to the external magnetic field, while the scalar magnetic potential inside each sphere is solved by imposing the exact boundary conditions known from the solution of the outside field. Finally, the numerical results with the given accuracy are generated from the expressions derived.

Keywords: Ferromagnetic spheres, scalar Laplace equation, scalar magnetic field, translational addition theorems.

INTRODUCTION

An exact analytical solution for the magnetic field inside ferromagnetic spheres of many-sphere systems is not available in literature. However the magnetic field inside and outside a single sphere was found in a previous study (Paris & Hurd, 1969). In this study, we solve the field equations for the magnetisation of systems of ferromagnetic spheres by imposing the corresponding boundary conditions. In order to obtain field problem solutions relative to many-sphere systems, the surfaces of the bodies involved have to be coordinated surfaces. To impose the boundary conditions at the surface of each sphere, we use the translational addition theorems (Cruzan, 1962) to express the field produced by the system in the coordinates system attached to a single sphere.

The external scalar magnetic field is taken to be uniform and the sphere system is placed in a homogeneous medium. In a homogeneous medium, the potentials outside and inside of ferromagnetic spheres satisfy the scalar Laplace equation. The scalar magnetic potential f can be expressed (Hayt & Buck, 2010) as

$$\mathbf{H} = -\nabla f \qquad ...(1)$$

where **H** is the magnetic field intensity and ∇ is the gradient operator. For ideal ferromagnetic bodies, the permeability of the material is linear and infinite. Therefore the scalar magnetic potential at the surface is constant.

Solution of the Laplace equation

The Laplace equation can be expressed as

$$\nabla^2 f(r, \theta, \varphi) = 0 \qquad ...(2)$$

where f is the scalar magnetic potential and ∇^2 is a Laplacian operator in spherical coordinates system. The general solution of equation (2) can be expressed in the form (Morse & Feshbach, 1953)

$$f = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} \left(C_{nm} r^{-(n+1)} + D_{nm} r^{n} \right) P_n^m(\cos\theta) \exp(-jm\varphi) \qquad ...(3)$$

where C_{nm} and D_{nm} are constants of integration, and m and n are integers. $P_n^m(\cos\theta)$ are associated Legendre functions (Smythe, 1968) of first kind of degree n and order m.

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Since the region is extended to infinity, the outside potential due to the sphere can be expressed as

$$\Phi(r,\theta,\varphi)=\sum_{n=0}^{\infty}\sum_{m=-n}^{n}C_{nm}r^{-(n+1)}P_n^m(\cos\theta)$$
$$\exp(-jm\varphi) \qquad \text{...(4)}$$

On the other hand, the inside potential is finite and can be expressed as

$$\Psi(r,\theta,\varphi)=\sum_{n=0}^{\infty}\sum_{m=-n}^{n}D_{nm}r^nP_n^m(\cos\theta)\exp(-jm\varphi) \qquad \text{...(5)}$$

Translational addition theorems for spherical scalar Laplacian functions

Figure 1 shows the translation of two spherical coordinate systems with the centers at O and O'. The translational addition theorems for spherical scalar wave functions in Cruzan (1962) can be simplified in the form (Ciric & Kotuwage, 2014) as

$$r^{-(n+1)}P_n^m(\cos\theta)\exp(-jm\varphi)=\begin{cases}\displaystyle\sum_{\nu=0}^{\infty}\sum_{\mu=-\nu}^{\nu}(-1)^{\mu+\nu}\frac{(n-m+\nu+\mu)!}{(n-m)!(\nu+\mu)!}\frac{(r')^{\nu}}{d^{\nu+n+1}}\\ \qquad\times P_{n+\nu}^{m-\mu}(\cos\theta_0)\exp\big(-j(m-\mu)\varphi_0\big)\\ \qquad\times P_\nu^\mu(\cos\theta')\exp(-j\mu\varphi'), \qquad r'\le d\\ \\ \displaystyle\sum_{\nu=0}^{\infty}\sum_{\mu=-\nu}^{\nu}(-1)^{\nu+\mu}\frac{(n-m+\nu+\mu)!}{(n-m)!(\nu+\mu)!}\frac{d^{\nu}}{(r')^{\nu+n+1}}\\ \qquad\times P_{n+\nu}^{m-\mu}(\cos\theta')\exp\big(-j(m-\mu)\varphi'\big)\\ \qquad\times P_\nu^\mu(\cos\theta_0)\exp(-j\mu\varphi_0), \qquad r'\ge d\end{cases} \qquad \text{...(6)}$$

$$r^nP_n^m(\cos\theta)\exp(-jm\varphi)=\sum_{\nu=0}^{n}\sum_{\mu=-\nu}^{\nu}\frac{(n+m)!}{(\nu+\mu)!(n+m-\nu-\mu)!}\frac{(r')^{\nu}}{d^{\nu+n}}P_{n-\nu}^{m-\mu}(\cos\theta_0)$$
$$\times P_\nu^\mu(\cos\theta')\exp\big(-j(m-\mu)\varphi_0\big)\exp(-j\mu\varphi'), \qquad r'\le d \qquad \text{...(7)}$$

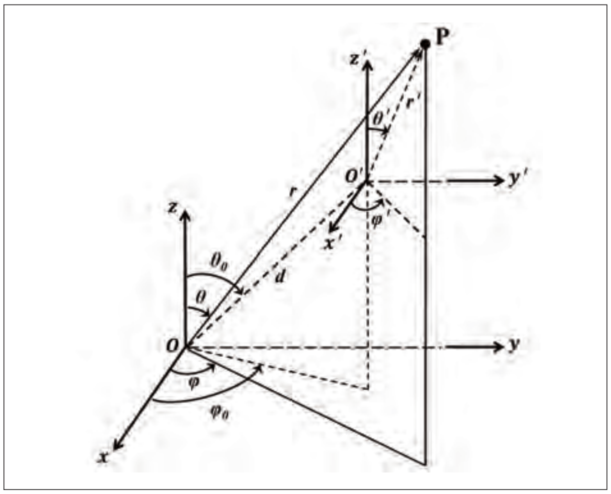

Figure 1: Coordinate translation from (r, θ, φ) to (r′, θ′, φ')

METHODS AND MATERIALS

A system of two ferromagnetic spheres of radii a_1 and a_2 with a distance d between their centers are placed in a homogeneous medium of permeability μ_0, as shown in Figure 2. (r_1, θ_1, φ_1) and (r_2, θ_2, φ_2) are the spherical coordinates attached to the sphere 1 and sphere 2, respectively. Let **P** be an arbitrary point outside the spheres.

The magnetic field along x axis

Consider the external field $\mathbf{H_0} = H_0\hat{x}$ when $\theta = \pi/2$. Then, the total scalar magnetic potential at **P** can be expressed as

$$\Phi^{tot}=\sum_{i=1}^{2}\Phi_i(r_i,\theta_i,\varphi_i)+\Phi_{ext(x)}(r_1,\theta_1,\varphi_1) \qquad \text{...(8)}$$

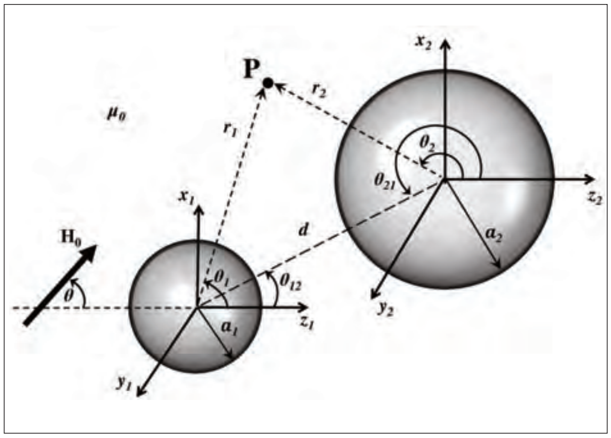

Figure 2: A two-sphere system with common *zx* planes placed in the presence of external magnetic field

where Φ_i is the potential due to the spheres i, with respect to its coordinates system. $\Phi_{ext(x)}$ is the potential due to the external field in spherical coordinates $(r_1, \theta_1, \varphi_1)$. The expression in equation (8) can be written in the coordinates system $(r_1, \theta_1, \varphi_1)$ as

$$\Phi^{tot}(r_1,\theta_1,\varphi_1) = \Phi_1(r_1,\theta_1,\varphi_1) + \Phi_2^{(1)}(r_1,\theta_1,\varphi_1) + \Phi_{ext(x)}(r_1,\theta_1,\varphi_1) \qquad ...(9)$$

where $\Phi_2^{(1)}(r_1, \theta_1, \varphi_1)$ is the potential due to the sphere 2 in coordinates system $(r_1, \theta_1, \varphi_1)$ attached to sphere 1. Taking the real part of equation (4), with $r = r_1$, $\theta = \theta_1$ and $\varphi = \varphi_1$ for sphere 1, we have

$$\Phi_1(r_1,\theta_1,\varphi_1) = \sum_{n=0}^{\infty}\sum_{m=-n}^{n} C_{nm}^{(1)}\left(\frac{a_1}{r_1}\right)^{(n+1)} P_n^m(\cos\theta_1)\cos m\varphi_1, \quad r_1 \geq a_1 \qquad ...(10)$$

where $C_{nm}^{(1)}$ is the constant of integration. Similarly, with $r = r_2$, $\theta = \theta_2$ and $\varphi = \varphi_2$ for sphere 2 as

$$\Phi_2(r_2,\theta_2,\varphi_2) = \sum_{q=0}^{\infty}\sum_{p=-q}^{q} C_{qp}^{(2)}\left(\frac{a_2}{r_2}\right)^{(q+1)} P_q^p(\cos\theta_2)\cos p\varphi_2, \quad r_2 \geq a_2 \qquad ...(11)$$

where $C_{qp}^{(2)}$ is the constant of integration.

Now, in order to find $\Phi_2^{(1)}(r_1, \theta_1, \varphi_1)$, we use the translational addition theorem in equation (6), with $r' = r_1$, $\theta' = \theta_1$, $\varphi' = \varphi_1$, $\theta_0 = (\pi - \theta_{12}) = \theta_{21}$ and $\varphi_0 = \varphi_{21}$, which gives

$$\Phi_2^{(1)}(r_1,\theta_1,\varphi_1) = \sum_{q=0}^{\infty}\sum_{p=-q}^{q}\sum_{\nu=0}^{\infty}\sum_{\mu=-\nu}^{\nu} C_{qp}^{(2)}(-1)^{\mu+\nu} \frac{(q-p+\nu+\mu)!}{(q-p)!(\nu+\mu)!}\left(\frac{a_2}{d}\right)^{q+1}\left(\frac{r_1}{d}\right)^{\nu} P_{q+\nu}^{p-\mu}(\cos\theta_{21}) P_\nu^\mu(\cos\theta_1)\cos\left((p-\mu)\varphi_{21} + \mu\varphi_1\right), \quad r_1 \leq d \qquad ...(12)$$

The potential corresponding to the external field is

$$\Phi_{ext(x)}(r_i,\theta_i,\varphi_i) = -H_0\, x_i + K_i \qquad i = 1,2 \qquad ...(13)$$

where K_i is a constant choosing $\Phi_{ext(x)} = 0$ at $x_1 = 0$ as reference, $K_1 = 0$ and $K_2 = -H_0\, d \sin\theta_{12} \cos\varphi_{12}$, and with $x_i = r_i \sin\theta_i \cos\varphi_i$ for $i = 1, 2$ we have

$$\Phi_{ext(x)}(r_1,\theta_1,\varphi_1) = -H_0\, r_1\, P_1^1(\cos\theta_1)\cos\varphi_1 \qquad ...(14a)$$

$$\Phi_{ext(x)}(r_2,\theta_2,\varphi_2) = -H_0\, r_2\, P_1^1(\cos\theta_2)\cos\varphi_2 - H_0\, d\sin\theta_{12}\cos\varphi_{12} \qquad ...(14b)$$

Then, by substituting equations (10), (12) and (14a) into equation (9), we can find the total magnetic potential outside the spheres.

Now by imposing boundary conditions we can find the above constants. The first condition is the total magnetic flux through the surface of each sphere is equal to zero, i.e.,

$$\oint_{S_i} B_{n_i}\, ds_{r_i} = 0, \qquad i = 1,2 \qquad ...(15)$$

where $B_{n_i} = -\mu_0 \frac{\partial}{\partial r_i}\{\Phi^{tot}\}\big|_{S_i}$ is the normal component of the magnetic flux density on the surface S_i of the sphere i with $ds_{r_i} = r_i^2 \sin\theta_i\, d\theta_i\, d\varphi_i$.

Substitute and apply the orthogonality of spherical harmonics (Smythe, 1968), we have

$$C_{00}^{(1)} = 0, \quad C_{00}^{(2)} = 0 \qquad ...(16)$$

The second condition that the magnetic potential at the surface of each sphere V_i, is constant, we have

$$\Phi^{tot}(r_i,\theta_i,\varphi_i)\Big|_{r_i=a_i} = V_i, \qquad i = 1,2. \qquad ...(17)$$

Substitute for each sphere and we have for sphere 1 as

$m = 0$

$$C_{n0}^{(1)} + \sum_{q=1}^{\infty}\sum_{p=-q}^{q} C_{qp}^{(2)}\,\zeta_2^{(1)}(p,q\mid 0,n\mid d,\theta_{21},\varphi_{21})\left(\frac{a_1}{d}\right)^{n} = 0, \quad n = 1,2,3,\ldots \qquad \text{...(18a)}$$

$m = 1$

$$C_{11}^{(1)} + \sum_{q=1}^{\infty}\sum_{p=-q}^{q} C_{qp}^{(2)}\,\zeta_2^{(1)}(p,q\mid 1,1\mid d,\theta_{21},\varphi_{21})\left(\frac{a_1}{d}\right) = -\frac{1}{2}a_1H_0, \quad n = 1 \qquad \text{...(18b)}$$

$$C_{n1}^{(1)} + \sum_{q=1}^{\infty}\sum_{p=-q}^{q} C_{qp}^{(2)}\,\zeta_2^{(1)}(p,q\mid 1,n\mid d,\theta_{21},\varphi_{21})\left(\frac{a_1}{d}\right)^{n} = 0, \quad n = 2,3,\ldots \qquad \text{...(18c)}$$

$m = -1$

$$C_{1,-1}^{(1)} + \sum_{q=1}^{\infty}\sum_{p=-q}^{q} C_{qp}^{(2)}\,\zeta_2^{(1)}(p,q\mid -1,1\mid d,\theta_{21},\varphi_{21})\left(\frac{a_1}{d}\right) = a_1H_0, \quad n = 1 \qquad \text{...(18d)}$$

$$C_{n,-1}^{(1)} + \sum_{q=1}^{\infty}\sum_{p=-q}^{q} C_{qp}^{(2)}\,\zeta_2^{(1)}(p,q\mid -1,n\mid d,\theta_{21},\varphi_{21})\left(\frac{a_1}{d}\right)^{n} = 0, \quad n = 2,3,\ldots \qquad \text{...(18e)}$$

where

$$\zeta_2^{(1)}\ (p,q\mid m,n\mid d,\theta_{21},\varphi_{21}) \equiv (-1)^{m+n}\frac{(q-p+n+m)!}{(q-p)!(n+m)!}\left(\frac{a_2}{d}\right)^{q+1} P_{q+n}^{p-m}(cos\ \theta_{21})\cos((p-m)\ \varphi_{21} + m\varphi_1)$$

and for sphere 2 as

$p = 0$

$$C_{q0}^{(2)} + \sum_{n=1}^{\infty}\sum_{m=-n}^{n} C_{nm}^{(1)}\,\zeta_1^{(2)}(m,n\mid 0,q\mid d,\theta_{12},\varphi_{12})\left(\frac{a_2}{d}\right)^{n_2} = 0, \quad q = 1,2,3,\ldots \qquad \text{...(19a)}$$

$p = 1$

$$C_{11}^{(2)} + \sum_{n=1}^{\infty}\sum_{m=-n}^{n} C_{nm}^{(1)}\,\zeta_1^{(2)}(m,n\mid 1,1\mid d,\theta_{12},\varphi_{12})\left(\frac{a_2}{d}\right) = -\frac{1}{2}a_2H_0, \quad q = 1 \qquad \text{...(19b)}$$

$$C_{q1}^{(2)} + \sum_{n=1}^{\infty}\sum_{m=-n}^{n} C_{nm}^{(1)}\,\zeta_1^{(2)}(m,n\mid 1,q\mid d,\theta_{12},\varphi_{12})\left(\frac{a_2}{d}\right)^{q} = 0, \quad q = 2,3,\ldots \qquad \text{...(19c)}$$

$p = -1$

$$C_{1,-1}^{(2)} + \sum_{n=1}^{\infty}\sum_{m=-n}^{n} C_{nm}^{(1)}\,\zeta_1^{(2)}(m,n\mid -1,1\mid d,\theta_{12},\varphi_{12})\left(\frac{a_2}{d}\right) = a_2H_0, \quad q = 1 \qquad \text{...(19d)}$$

$$C_{q,-1}^{(2)} + \sum_{n=1}^{\infty}\sum_{m=-n}^{n} C_{nm}^{(1)}\,\zeta_1^{(2)}(m,n\mid -1,q\mid d,\theta_{12},\varphi_{12})\left(\frac{a_2}{d}\right)^{q} = 0, \quad q = 2,3,\ldots \qquad \text{...(19e)}$$

where

$$\zeta_1^{(2)}\ (m,n\mid p,q\mid d,\theta_{12},\varphi_{12}) \equiv (-1)^{p+q}\frac{(n-m+q+p)!}{(n-m)!(q+p)!}\left(\frac{a_1}{d}\right)^{n+1} P_{n+q}^{m-p}(cos\ \theta_{12})\cos((m-p)\varphi_{12})$$

$C_{nm}^{(1)} = 0$ and $C_{qp}^{(2)} = 0$ for all $m, p \neq 0, +1, -1$.

To obtain numerical solutions, the infinite set of linear equations in (18) and (19), which satisfy the constants of integration $C_{nm}^{(1)}$ and $C_{qp}^{(2)}$ are to be truncated to a finite number of terms N as follows. For m, $p = 0, +1, -1$ and $n, q = 1 \ldots N$, denoting

$$S_{q,p}^{(n,m)} = \zeta_2^{(1)}(p,q\mid m,n\mid d,\theta_{21},\varphi_{21})\left(\frac{a_1}{d}\right)^{n},$$

$$T_{n,m}^{(q,p)} = \zeta_1^{(2)}(m,n\mid p,q\mid d,\theta_{12},\varphi_{12})\left(\frac{a_2}{d}\right)^{q},$$

the finite system of equations truncated to $6N \times 6N$ matrix and then the system can be written in form as

$$\begin{pmatrix} [I]_{3N\times 3N} & [S]_{3N\times 3N} \\ [T]_{3N\times 3N} & [I]_{3N\times 3N} \end{pmatrix}_{6N\times 6N} \begin{pmatrix} [C^{(1)}]_{3N\times 1} \\ [C^{(2)}]_{3N\times 1} \end{pmatrix}_{6N\times 1} = H_0\begin{pmatrix} [A^{(1)}]_{3N\times 1} \\ [A^{(2)}]_{3N\times 1} \end{pmatrix}_{6N\times 1} \qquad \text{...(20)}$$

where

$$[I]_{3N\times 3N} = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix} \quad \text{...(21)}$$

$$[S]_{3N\times 3N} = \begin{pmatrix} [S_{q,p}^{(n,0)}]_{N\times N} & [Z]_{N\times N} & [Z]_{N\times N} \\ [Z]_{N\times N} & [S_{q,p}^{(n,1)}]_{N\times N} & [Z]_{N\times N} \\ [Z]_{N\times N} & [Z]_{N\times N} & [S_{q,p}^{(n,-1)}]_{N\times N} \end{pmatrix} \quad \text{...(22)}$$

$$[T]_{3N\times 3N} = \begin{pmatrix} [T_{n,m}^{(q,0)}]_{N\times N} & [Z]_{N\times N} & [Z]_{N\times N} \\ [Z]_{N\times N} & [T_{n,m}^{(q,1)}]_{N\times N} & [Z]_{N\times N} \\ [Z]_{N\times N} & [Z]_{N\times N} & [T_{n,m}^{(q,-1)}]_{N\times N} \end{pmatrix} \quad \text{...(23)}$$

$$[C^{(1)}]_{3N\times 1} = \left(C_{10}^{(1)}\, C_{20}^{(1)} \ldots C_{N0}^{(1)}\, C_{11}^{(1)}\, C_{21}^{(1)} \ldots C_{N1}^{(1)}\, C_{1,-1}^{(1)}\, C_{2,-1}^{(1)} \ldots C_{N,-1}^{(1)}\right)^t \quad \text{...(24)}$$

$$[C^{(2)}]_{3N\times 1} = \left(C_{10}^{(2)}\, C_{20}^{(2)} \ldots C_{N0}^{(2)}\, C_{11}^{(2)}\, C_{21}^{(2)} \ldots C_{N1}^{(2)}\, C_{1,-1}^{(2)}\, C_{2,-1}^{(2)} \ldots C_{N,-1}^{(2)}\right)^t \quad \text{...(25)}$$

$$[A^{(1)}]_{3N\times 1} = \left(0 \;\cdots\; 0 \;\; -\tfrac{a_1}{2} \;\; 0 \;\cdots\; 0 \;\; a_1 \;\; 0 \;\cdots\; 0\right)^t \quad \text{...(26)}$$

$$[A^{(2)}]_{3N\times 1} = \left(0 \;\cdots\; 0 \;\; -\tfrac{a_2}{2} \;\; 0 \;\cdots\; 0 \;\; a_2 \;\; 0 \;\cdots\; 0\right)^t \quad \text{...(27)}$$

and $[Z]_{N\times N}$ is a zero matrix.

After solving the matrix in equation (20), we can find the potential outside the spheres either from equation (8) or in coordinates system $(r_1, \theta_1, \varphi_1)$ using equation (9). Then the magnetic field intensity **H** outside can be calculated from $\mathbf{H} = -\nabla\, \Phi^{tot}$ in one of the coordinates system as

$$\mathbf{H}(r_i, \theta_i, \varphi_i) = H_{r_i}\hat{r}_i + H_{\theta_i}\hat{\theta}_i + H_{\varphi_i}\hat{\varphi}_i, \qquad i = 1, 2 \quad \text{...(28)}$$

where

$$H_{r_i} = -\frac{\partial}{\partial r_i}\left\{\Phi^{tot}\right\}, \qquad H_{\theta_i} = -\frac{1}{r_i}\frac{\partial}{\partial \theta_i}\left\{\Phi^{tot}\right\};$$

$$H_{\varphi_i} = -\frac{1}{r_i \sin\theta_i}\frac{\partial}{\partial \varphi_i}\left\{\Phi^{tot}\right\}.$$

Now consider the field quantities inside the spheres. The potential inside the spheres Ψ can be expressed by taking the real part of equation (5) in their coordinates system as

$$\Psi^{(1)} = \sum_{n=0}^{\infty}\sum_{m=-n}^{n} D_{nm}^{(1)}\left(\frac{r_1}{a_1}\right)^n P_n^m(\cos\theta_1)$$

$$\cos(m\varphi_1), \quad r_1 < a_1 \quad \text{...(29a)}$$

$$\Psi^{(2)} = \sum_{q=0}^{\infty}\sum_{p=-q}^{q} D_{qp}^{(2)}\left(\frac{r_2}{a_2}\right)^q P_q^p(\cos\theta_2)$$

$$\cos(p\varphi_2), \quad r_2 < a_2 \quad \text{...(29b)}$$

where $D_{nm}^{(1)}$ and $D_{qp}^{(2)}$ are constants of integration. To find these constants we can use a boundary condition that the normal component of the total magnetic flux density outside is continuous across the surface of each sphere. This can be expressed for each sphere as

$$-\frac{\partial}{\partial r_i}\left\{\Psi^{(i)}\right\}\Big|_{r_i=a_i} = -\mu_0\frac{\partial}{\partial r_i}\left\{\Phi^{tot}(r_i, \theta_i, \varphi_i)\right\}\Big|_{r_i=a_i},$$

$$i = 1, 2 \quad \text{...(30)}$$

Simplify by using the orthogonality of spherical harmonics to obtain for sphere 1 as

$$D_{n0}^{(1)} = \mu_0\frac{(2n+1)}{n}\sum_{q=1}^{\infty}\sum_{p=-1}^{1} C_{qp}^{(2)}\left(\frac{a_1}{d}\right)^n$$

$$\zeta_2^{(1)}(p, q \mid 0, n \mid d, \theta_{21}, \varphi_{21}), \qquad n = 1, 2, 3, \ldots \quad \text{...(31a)}$$

$$D_{11}^{(1)} = 3\mu_0\left\{\sum_{q=1}^{\infty}\sum_{p=-1}^{1} C_{qp}^{(2)}\left(\frac{a_1}{d}\right)\right.$$

$$\left.\zeta_2^{(1)}(p, q \mid 1, 1 \mid d, \theta_{21}, \varphi_{21}) + \frac{1}{2}a_1 H_0\right\}, \quad n = 1 \quad \text{...(31b)}$$

$$D_{n1}^{(1)} = \mu_0\frac{(2n+1)}{n}\sum_{q=1}^{\infty}\sum_{p=-1}^{1} C_{qp}^{(2)}\left(\frac{a_1}{d}\right)^n$$

$$\zeta_2^{(1)}(p, q \mid 1, n \mid d, \theta_{21}, \varphi_{21}), \qquad n = 2, 3, \ldots \quad \text{...(31c)}$$

$$D_{1,-1}^{(1)} = 3\mu_0\left\{\sum_{q=1}^{\infty}\sum_{p=-1}^{1} C_{qp}^{(2)}\left(\frac{a_1}{d}\right)\right.$$

$$\left.\zeta_2^{(1)}(p, q \mid -1, 1 \mid d, \theta_{21}, \varphi_{21}) - a_1 H_0\right\}, \quad n = 1 \quad \text{...(31d)}$$

$$D_{n,-1}^{(1)} = \mu_0\frac{(2n+1)}{n}\sum_{q=1}^{\infty}\sum_{p=-1}^{1} C_{qp}^{(2)}\left(\frac{a_1}{d}\right)^n$$

$$\zeta_2^{(1)}(p, q \mid -1, n \mid d, \theta_{21}, \varphi_{21}), \quad n = 2, 3, \ldots \quad \text{...(31e)}$$

and for sphere 2 as

$$D^{(2)}_{q0} = \mu_0 \frac{(2q+1)}{q} \sum_{n=1}^{\infty} \sum_{m=-1}^{1} C^{(1)}_{nm} \left(\frac{a_2}{d}\right)^q \zeta_1^{(2)}(m,n \mid 0,q \mid d,\theta_{12},\varphi_{12}), \quad q = 1,2,3,\ldots \qquad \text{...(32a)}$$

$$D^{(2)}_{11} = 3\mu_0 \left\{ \sum_{n=1}^{\infty} \sum_{m=-1}^{1} C^{(1)}_{nm} \left(\frac{a_2}{d}\right) \zeta_1^{(2)}(m,n \mid 1,1 \mid d,\theta_{12},\varphi_{12}) + \frac{1}{2} a_2 H_0 \right\}, \quad q = 1 \qquad \text{...(32b)}$$

$$D^{(2)}_{q1} = \mu_0 \frac{(2q+1)}{q} \sum_{n=1}^{\infty} \sum_{m=-1}^{1} C^{(1)}_{nm} \left(\frac{a_2}{d}\right)^q \zeta_1^{(2)}(m,n \mid 1,q \mid d,\theta_{12},\varphi_{12}), \quad q = 2,3,\ldots \qquad \text{...(32c)}$$

$$D^{(2)}_{1,-1} = 3\mu_0 \left\{ \sum_{n=1}^{\infty} \sum_{m=-1}^{1} C^{(1)}_{nm} \left(\frac{a_2}{d}\right) \zeta_1^{(2)}(m,n \mid -1,1 \mid d,\theta_{12},\varphi_{12}) - a_2 H_0 \right\}, \quad q = 1 \qquad \text{...(32d)}$$

$$D^{(2)}_{q,-1} = \mu_0 \frac{(2q+1)}{q} \sum_{n=1}^{\infty} \sum_{m=-1}^{1} C^{(1)}_{nm} \left(\frac{a_2}{d}\right)^q \zeta_1^{(2)}(m,n \mid -1,q \mid d,\theta_{12},\varphi_{12}), \quad q = 2,3,\ldots \qquad \text{...(32e)}$$

with $D^{(1)}_{nm}, D^{(2)}_{qp} = 0$ for $m, p \neq 0, +1, -1$.

Once we determine these constants $D_{nm}^{(1)}$ and $D_{qp}^{(2)}$ in equation (29), we can find the potential inside each sphere. Then the magnetic flux density **B** inside can be calculated from $\mathbf{B} = -\nabla \Psi$ as

$$\mathbf{B}^{(i)}(r_i, \theta_i, \varphi_i) = B_{r_i} \hat{r_i} + B_{\theta_i} \hat{\theta_i} + B_{\varphi_i} \hat{\varphi_i}, \quad i = 1, 2 \qquad \text{...(33)}$$

where

$$B_{r_i} = -\frac{\partial}{\partial r_i}\left\{\Psi^{(i)}\right\}, \qquad B_{\theta_i} = -\frac{1}{r_i}\frac{\partial}{\partial \theta_i}\left\{\Psi^{(i)}\right\},$$

$$B_{\varphi_i} = -\frac{1}{r_i \sin\theta_i}\frac{\partial}{\partial \varphi_i}\left\{\Psi^{(i)}\right\}.$$

The magnetic field along z axis

Now we consider the external field $\mathbf{H_0} = H_0\,\hat{z}$ when $\theta = 0$ (Figure 2). As previous, the total scalar magnetic potential outside can be expressed in $(r_1, \theta_1, \varphi_1)$ as

$$\Phi^{tot}(r_1, \theta_1, \varphi_1) = \Phi_1 + \Phi_2^{(1)} + \Phi_{ext(z)} \qquad \text{...(34)}$$

where $\Phi_{ext(z)}$ is the potential due to the external field.

The outside potential due to the sphere can be expressed from equation (4) with $r = r_1$, $\theta = \theta_1$ and $\varphi = \varphi_1$ for sphere 1, we have

$$\Phi_1 = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} C^{(1)}_{nm} \left(\frac{a_1}{r_1}\right)^{n+1} P_n^m(\cos\theta_1) \exp(-jm\varphi_1), \qquad r_1 \geq a_1 \qquad \text{...(35)}$$

where $C^{(1)}_{nm}$ is the constant of integration.

In order to find $\Phi_2^{(1)}$, we use the translational addition theorem in equation (6) as previously, which gives

$$\Phi_2^{(1)} = \sum_{q=0}^{\infty} \sum_{p=-q}^{q} \sum_{\nu=0}^{\infty} \sum_{\mu=-\nu}^{\nu} C^{(2)}_{qp} (-1)^{\mu+\nu} \frac{(q-p+\nu+\mu)!}{(q-p)!(\nu+\mu)!} \left(\frac{a_2}{d}\right)^{q+1} \left(\frac{r_1}{d}\right)^{\nu} P_{q+\nu}^{p-\mu}(\cos\theta_{21})\, P_\nu^\mu(\cos\theta_1) \exp\left(-j(p-\mu)\varphi_{21}\right) \exp(-j\mu\varphi_1), \qquad r_1 \leq d \qquad \text{...(36)}$$

where $C^{(2)}_{qp}$ is the constant of integration.

The external field is given by $\mathbf{H_0} = -\nabla \Phi_{ext(z)}$ and, thus,

$$\Phi_{ext(z)}(r_1, \theta_1) = -H_0 z_1 + K_1 \qquad \text{...(37a)}$$

$$\Phi_{ext(z)}(r_2, \theta_2) = -H_0 z_2 + K_2 \qquad \text{...(37b)}$$

where the constants K_1 and K_2 are used to fix the reference potential. We choose $\Phi_{ext(z)} = 0$ at $z_1 = 0$ and, then, $K_1 = 0$ and $K_2 = -H_0 d$, and with $z_i = r_i \cos\theta_i$ for $i = 1, 2$, which yields

$$\Phi_{ext(z)}(r_1, \theta_1) = -H_0 r_1 P_1(\cos\theta_1) \qquad \text{...(38a)}$$

$$\Phi_{ext(z)}(r_2, \theta_2) = -H_0 r_2 P_1(\cos\theta_2) - H_0 d \qquad \text{...(38b)}$$

where P_1 is Legendre polynomial of degree 1.

Then, by substituting equations (35), (36) and (38a) into equation (34), we can find the total magnetic potential outside the spheres. To find the constants in equation (34), we impose two boundary conditions as in equation (15) and (17). As previously, then we can obtain the set of an infinite system of linear algebraic equations (Anthonys, 2014) satisfied by constants $C_{nm}^{(1)}$ and $C_{qp}^{(2)}$, only for m, $p = 0$. The constants of integration $C_{nm}^{(1)} = 0$ and $C_{qp}^{(2)} = 0$ for m, $p \neq 0$. For the numerical solutions this infinite system can be truncated to a $2N \times 2N$ matrix. By solving this matrix, we can find the total magnetic potential and the magnetic field intensity outside by using equations (34) and (28), respectively.

On the other hand, the magnetic potential inside Ψ can be expressed by using equation (5) with $m = 0$, in their respective coordinates system, we have

$$\Psi^{(1)}(r_1,\theta_1) = \sum_{n=0}^{\infty} D_{n0}^{(1)} \left(\frac{r_1}{a_1}\right)^n P_n(cos\,\theta_1), \qquad r_1 < a_1 \qquad ...(39a)$$

$$\Psi^{(2)}(r_2,\theta_2) = \sum_{q=0}^{\infty} D_{q0}^{(2)} \left(\frac{r_2}{a_2}\right)^q P_q(cos\,\theta_2), \qquad r_2 < a_2 \qquad ...(39b)$$

where $D_{n0}^{(1)}$ and $D_{q0}^{(2)}$ are constants of integration. To find these constants we impose the boundary condition as in equation (30) and simplify by using the orthogonality of Legendre polynomials (Smythe, 1968). Then the magnetic flux density inside **B** each sphere can be computed by using equation (33) in their attached coordinates system.

RESULTS

Figure 3 shows two spheres of the same radius a, with the gap g in the presence of the magnetic field $\mathbf{H_0}\angle\theta$.

Using equation (28), numerical values for magnetic field intensity in $(r_1, \theta_1, \varphi_1)$ were generated at the points $\mathbf{P_1}$, $\mathbf{P_2}$, $\mathbf{P_3}$ in the zx plane for different ratios g/a when the external field is oriented along the x axis. The results have been tabulated in Table 1. The infinite system of linear equations in (18) and (19) has been truncated by retaining 45 coefficients of each $C_{nm}^{(1)}$ and $C_{qp}^{(2)}$ (excluding $C_{00}^{(1)} = C_{00}^{(2)} = 0$), which are computed using the matrix equation in (20), with 90 unknown constants. Table 2 shows the numerical values for the magnetic flux density inside of each sphere obtained by using equation (33). The expressions in equations (31) and (32) have been truncated each to the same number of coefficients, i.e., 45, for $D_{nm}^{(1)}$ and for $D_{qp}^{(2)}$ (excluding $D_{00}^{(1)} = D_{00}^{(2)} = 0$).

Table 1: Magnetic field intensity at $\mathbf{P_1}$, $\mathbf{P_2}$, $\mathbf{P_3}$ for various ratios g/a when $\theta = \pi / 2$ with $\theta_{12} = 30^0$, $\varphi_{12} = 0^0$, $H_0 = 1$A/m, $N = 15$

External field	Point	Magnetic field intensity H (A/m) for various ratios g/a								
		100.0	10.00	2.000	1.000	0.500	0.100	0.050	0.010	0.005
H_x	$\mathbf{P_1}$	1.8028	1.7964	1.7174	1.6757	1.6628	1.6777	1.6796	1.6823	1.6827
	$\mathbf{P_2}$	1.8028	1.8131	2.3280	3.5509	5.7204	10.2037	11.2895	12.2300	12.3544
	$\mathbf{P_3}$	1.0003	1.1763	4.9945	7.7284	9.8172	11.6563	12.0964	12.4040	12.4422

Table 2: Magnetic flux density inside each sphere when $\theta = \pi / 2$ with $a = 2$ nm, $\theta_{12} = 30^0$, $\varphi_{12} = 0^0$, $g = 2$ nm, $H_0 = 1$A/m, $N = 15$

External field	Point inside		Magnetic flux density $B^{(i)}$ (T)	
	r_i (nm)	θ_i (deg)	$B^{(1)}$	$B^{(2)}$
H_x	0.20	-90.00	5.1888E-06	5.1889E-06
	0.75	-90.00	5.2007E-06	5.1819E-06
	1.00	-30.00	2.4777E-06	2.6780E-06
	1.00	60.00	4.4812E-06	4.5137E-06
	1.50	90.00	5.2235E-06	5.1682E-06
	1.90	150.00	2.2906E-06	2.7658E-06

Table 3: Magnetic field intensity at $\mathbf{P_1}$, $\mathbf{P_2}$, $\mathbf{P_3}$ for various ratios g/a when $\theta = 0$ with $\theta_{12} = 30^0$, $\varphi_{12} = 0^0$, $H_0 = 1$A/m

External field	Point	Magnetic field intensity H (A/m) for various ratios g/a								
		100.0	10.00	2.000	1.000	0.500	0.100	0.050	0.010	0.005
H_z	$\mathbf{P_1}$	2.7839	2.7857	2.8233	2.8694	2.9246	3.0158	3.0334	3.0490	3.0511
	$\mathbf{P_2}$	2.5981	2.6007	2.7060	2.9400	3.3926	4.7061	5.0747	5.4518	5.5058
	$\mathbf{P_3}$	2.5981	2.6007	2.7060	2.9400	3.3926	4.7061	5.0747	5.4518	5.5058

Table 3 shows the results with 5-digit accuracy for the magnetic field intensity at the points $\mathbf{P_1}$, $\mathbf{P_2}$, $\mathbf{P_3}$ for various gap ratios g/a when the external field is oriented along the z axis (Figure 3). These numerical values were generated by using equations (34) and (28). On the other hand, the magnetic flux density inside of each sphere is tabulated in Table 4. These values were computed by using equation (33).

Table 5 shows the magnetic field intensities at the three points for different H_0, along the x axis and z axis. These values are generated for the external field in the range ($0.001 \leq H_0 \leq 10$). Figures 4(a) and 4(b) show the corresponding graphs, respectively. For all numerical calculations, the permeability of the medium outside was taken to be that of free space, i.e., $\mu_0 = 4\pi \times 10^{-7}$ H/m. Figure 5 shows the corresponding field lines for the common zx planes, when the external field is oriented along the x axis.

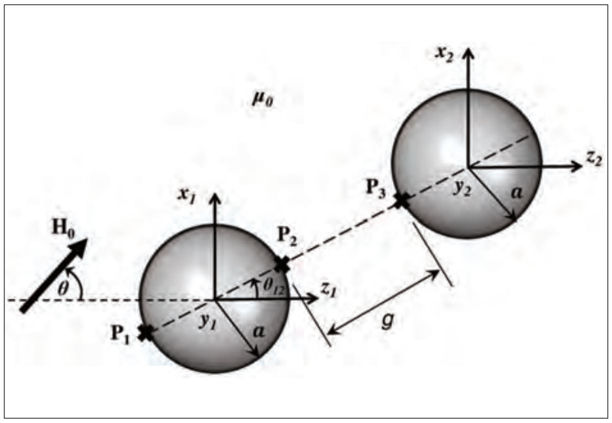

Figure 3: A similar two ferromagnetic spheres with common zx planes in an external magnetic field

Table 4: Magnetic flux density inside each sphere when $\theta = 0$ with $a = 2$ nm, $\theta_{12} = 30^0$, $\varphi_{12} = 0^0$, $g = 2$ nm, $H_0 = 1$A/m, $N = 15$

External field	Point inside		Magnetic flux density $B^{(i)}$ (T)	
	r_i (nm)	θ_i (deg)	$B^{(1)}$	$B^{(2)}$
H_z	0.20	-90.00	3.9530E-06	3.9541E-06
	0.75	-90.00	3.9464E-06	3.9601E-06
	1.00	-30.00	4.0082E-06	3.9011E-06
	1.00	60.00	3.9618E-06	3.9449E-06
	1.50	90.00	3.9315E-06	3.9727E-06
	1.90	150.00	3.9464E-06	3.9578E-06

Table 5: Magnetic field intensity at $\mathbf{P_1}$, $\mathbf{P_2}$, $\mathbf{P_3}$ for different H_0 when $\theta = \pi/2$, 0 with $a = 2$ nm, $\theta_{12} = 30^0$, $\varphi_{12} = 0^0$, $g = 2$ nm, $N = 15$

	Magnetic field intensity H (A/m) for different H_0 when the field is oriented					
	along x axis ($\theta = \pi/2$)			along z axis ($\theta = 0$)		
H_0 (A/m)	$\mathbf{P_1}$	$\mathbf{P_2}$	$\mathbf{P_3}$	$\mathbf{P_1}$	$\mathbf{P_2}$	$\mathbf{P_3}$
0.001	0.0017	0.0036	0.0077	0.0029	0.0029	0.0029
0.005	0.0084	0.0178	0.0386	0.0143	0.0147	0.0147
0.010	0.0168	0.0355	0.0773	0.0287	0.0294	0.0294
0.500	0.8378	1.7755	3.8642	1.4347	1.4700	1.4700
1.000	1.6757	3.5509	7.7284	2.8694	2.9400	2.9400
2.000	3.3513	7.1018	15.4568	5.7388	5.8799	5.8799
5.000	8.3784	17.7545	38.6421	14.3471	14.6998	14.6998
10.000	16.7567	35.509	77.2842	28.6942	29.3997	29.3997

DISCUSSION AND CONCLUSION

In this study, the exact analytical expressions were derived regarding the magnetisation of ferromagnetic spheres in the presence of external magnetic fields on the basis of the exact field equations and by imposing the boundary conditions. In the case of the geometry of zx plane, the general expressions for the set of linear equations are simple as in equations (18) - (19). As a result, the generated matrices for such geometries are relatively simple, which facilitates the computation of the numerical results.

The results are given for the points when the coordinate $\varphi = 0$ (i.e., on the zx plane), as shown in Figure 3. The field quantities for the system can be computed by the superposition of the field quantities due to the two spheres and to the external field, using either the attached coordinate systems (with sets of single series) or a single coordinate system (with sets of multiple series, after using the translational addition theorems). In the former case, the coordinate relations between the two coordinate systems are needed.

It should be noted that numerical values of the field quantities were not exactly the same when using the two procedures. This is due to the summation of the terms in the respective series. For example, let us consider N terms in the truncated single series. When using the translation to only one coordinate system, the number of terms in the translated portion of each expression is N^2, which increases usually the computational errors. Thus, it is expected that the expressions containing single series yield more accurate numerical results than the expressions containing multiple series.

The value of N in the system is varied in order to obtain a certain accuracy and it increases when the relative gap ratio g/a decreases. As an example, when $g/a = 1$, then, $N = 10$ terms, when $g/a = 0.1$ or 0.05, then $N = 20$ terms, and $N = 40$ when $g/a = 0.005$ are necessary to have an accuracy of 5-digit when the magnetic field is along the z axis.

The magnetic field intensities (also magnetic flux densities) outside the spheres depend on the size of the spheres and the relative distance between the spheres. From the results (Tables 1 and 3) we can conclude that the magnetic field intensity increases when the spheres are close to each other. Then the interaction forces increase with high values. As a result, the magnetic flux density inside (magnetisation) the spheres also increase. As seen in Table 5 it is noted that the field intensities at the three points also increase proportionately when the external field increases. This is clearly shown in Figure 4.

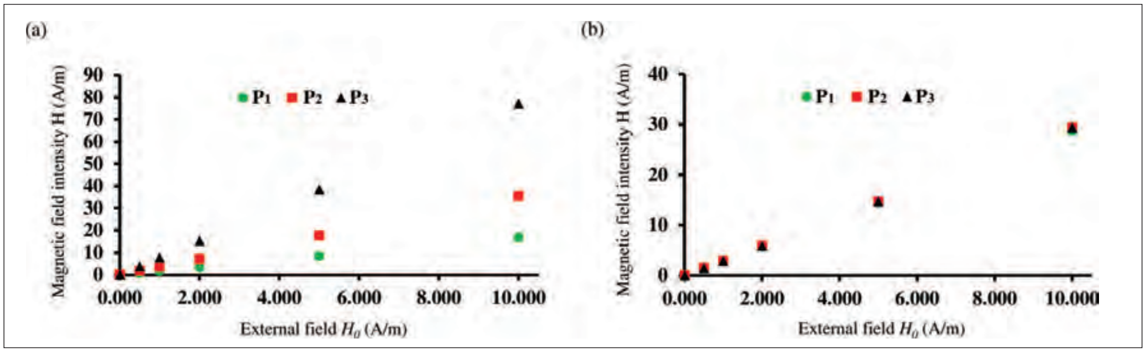

Figure 4: Magnetic field intensities at $\mathbf{P}_1$, $\mathbf{P}_2$, $\mathbf{P}_3$ for different H_0 with $a = 2$ nm, $\theta_{12} = 30^0$, $\varphi_{12} = 0^\circ$, $g = 2$ nm, $N = 15$: (a) when $\theta = \pi/2$; (b) when $\theta = 0$

Since the scalar magnetic potential at the surface is constant (the permeability of ideal ferromagnetic materials is linear and infinite), the field lines are perpendicular to the surface of each sphere (Figure 5). The analysis of the field outside the ferromagnetic spheres allows the evaluation of the intensification of the field, while the field inside determines the magnetisation of the spheres.

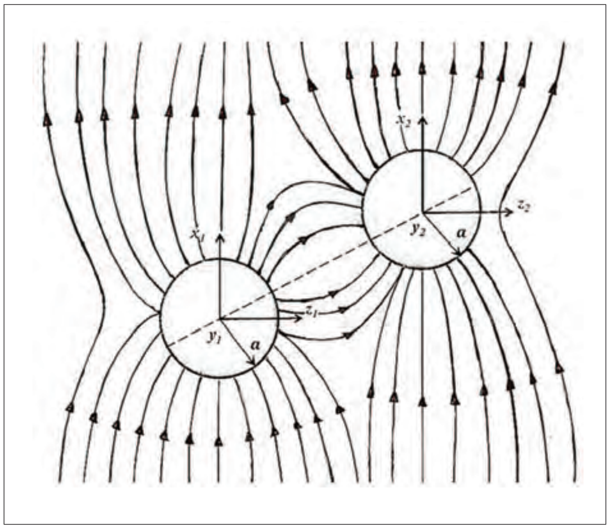

Figure 5: Magnetic field lines for the common zx planes when $\theta = \pi/2$

The investigation of the behaviour of such ferromagnetic systems is useful for the construction of various models to be employed, for instances, in nano-scale engineering applications (Barbic & Scherer, 2006) and in ferrohydrodynamic applications (Rosensweig, 1985). The construction and the solution presented in this article can be used when the scalar magnetic potential of the field under consideration satisfies the Laplace equation and both Dirichlet and Neumann boundary conditions (Morse & Feshbach, 1953). The benchmark numerical results generated in this study are valuable as reference data to inspect the accuracy of different numerical methods developed to solve magnetostatic boundary value problems in the presence of external magnetic fields for real world applications.

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